ELSEVIER

Available online at www.sciencedirect.com
ScienceDirect

Topology and its Applications 154 (2007) 1178–1194

Topology and its Applications

www.elsevier.com/locate/topol

A note on the homotopy theory of pro-G-spectra

Steven R. Costenoble *, Stefan Waner

Department of Mathematics, 103 Hofstra University, Hempstead, NY 11549-1030, USA

Received 26 June 2006; received in revised form 30 November 2006; accepted 4 December 2006

Abstract

This paper represents a step toward a model structure on pro-spectra in which the weak equivalences are the maps inducing pro-isomorphisms of all pro-homotopy groups. We construct a category in which these weak equivalences are inverted and show that we have not inverted "too much," in the sense that isomorphic objects still give pro-isomorphic cohomology groups.

MSC: primary 55P42; secondary 18E35, 55N91, 55P60, 55P91, 55U35

Keywords: Equivariant; Pro-spectrum; Weak equivalence; Pro-homotopy

0. Introduction

It would be very useful to have a model structure on the category of pro-spectra in which the weak equivalences were the maps inducing pro-isomorphisms of all pro-homotopy groups. Such model structures are known for pro-spaces; for example Grossman constructed one for towers of spaces [5], which Isaksen generalized to general pro-spaces [6]. However, no such model structure on pro-spectra is yet known, nor do we give one here. Edwards and Hastings [3] did give a model structure on the category of pro-spectra, but they used essentially levelwise weak equivalences, a stronger notion than the one above. (Isaksen [7] extended their construction to more general pro-categories.) Fausk and Isaksen [4] give a model structure on pro-spectra in which a weak equivalence is a map that induces pro-isomorphisms of all pro-homotopy groups but must also be an essentially levelwise m-equivalence for some m, hence again these weak equivalences are stronger than would be ideal.

This paper represents a step toward the model structure we desire, and supplies justification for claims we make in [2]. We work with towers of G-spectra, which we call tow-G-spectra, and construct a homotopy category in which weak equivalences are inverted. Although this may not be the localization at the weak equivalences, we do show that towers that are equivalent in this category have isomorphic pro-homotopy groups and give isomorphic pro-cohomology theories.

To construct this homotopy category we work in a category of ind-pro-G-spectra, namely the full subcategory generated by levelwise sequences of towers of G-spectra, which we call seq-tow-G-spectra. We define a notion of

* Corresponding author.
E-mail addresses: steven.r.costenoble@hofstra.edu (S.R. Costenoble), stefan.waner@hofstra.edu (S. Waner).

0166-8641/$ – see front matter © 2007 Elsevier B.V. All rights reserved.
doi:10.1016/j.topol.2006.12.001

weak equivalence in this category, generalizing weak equivalence of towers, and show that we can localize at these weak equivalences (without defining a full model structure).

Key steps in our arguments involve constructing maps between certain tow-spectra, requiring in turn the solution of multiple lifting problems, so we show in Section 1 that we can specialize to *fibrant* tow-G-spectra, the tow-G-spectra for which these lifting problems can automatically be solved. The main result in this section is that every tow-G-spectrum can be replaced by a fibrant tow-G-spectrum. Along the way, we introduce an interesting notion of well-behaved pairs of inverse homotopy equivalences we refer to as *adjoint homotopy equivalences* (Definition 1.9).

We invert the weak equivalences by constructing a class of cell complexes and then using cellular approximation together with fibrant approximation to give us the required localization. The class of cell complexes we use is described in Section 3, and our cellular approximation result is Theorem 3.4.

In Section 4 we describe the weak equivalences between seq-tow-G-spectra, prove a Whitehead Theorem, and also our localization result (Theorem 4.10). Finally, we show that equivalent tow-G-spectra have isomorphic equivariant pro-homotopy groups and determine isomorphic pro-cohomology theories.

1. Fibrations

We work in the full subcategory of pro-$G\mathscr{S}$, the category of pro-G-spectra, consisting of all pro-spectra isomorphic to towers. This subcategory, tow-$G\mathscr{S}$, includes any pro-G-spectrum whose indexing set has a countable cofinal subset. We shall use the notation $X[*]$ to denote a pro- or tow-G-spectrum, whose individual G-spectra are the $X[i]$ as i ranges over the indexing set. For towers, we take the indexing set to be the non-negative integers, with structure maps $X[i] \to X[i-1]$.

Definition 1.1. A map $f : X[*] \to Y[*]$ of pro-G-spectra is a *weak equivalence* if, for each integer n, the induced map $f_* : \bar{\pi}_n X[*] \to \bar{\pi}_n Y[*]$ is a pro-isomorphism of pro-Mackey functors. Here, $\bar{\pi}_n Z$ is the Mackey functor defined by $\bar{\pi}_n Z(G/H) = \pi_n^H Z = \pi_n(Z^H)$.

Definition 1.2. A map $X[*] \to Y[*]$ of tow-G-spectra is a *fibration* if it has a levelwise representative in which each map $X[i] \to X[i-1] \times_{Y[i-1]} Y[i]$ is a Hurewicz fibration of G-spectra. A tow-G-spectrum $X[*]$ is *fibrant* if the map $X[*] \to *$ is a fibration. Equivalently, each map $X[i] \to X[i-1]$ is a fibration of G-spectra.

We first need the following facts about fibrations.

Lemma 1.3. *If $X[*] \to Y[*]$ is a fibration, then it has the homotopy lifting property, meaning that, for any pro-G-spectrum $A[*]$ we can always find a lift in the following diagram*:

$$\begin{tikzcd} A[*] \arrow[r] \arrow[d, "i_0"'] & X[*] \arrow[d] \\ A[*] \wedge I_+ \arrow[r] \arrow[ur, dashed] & Y[*] \end{tikzcd}$$

Proof. We construct the lift by induction. The inductive step requires a lift in the following diagram:

$$\begin{tikzcd} A[j] \arrow[r] \arrow[d, "i_0"'] & X[i] \arrow[d] \\ A[j] \wedge I_+ \arrow[r] \arrow[ur, dashed] & X[i-1] \times_{Y[i-1]} Y[i] \end{tikzcd}$$

The map on the right is a fibration by definition, hence we can always find such a lift. □

Lemma 1.4. *If $X[*] \to Y[*]$ is a fibration then, for every i, the maps $X[i] \to Y[i]$ and $X[i-1] \times_{Y[i-1]} Y[i] \to Y[i]$ are fibrations of G-spectra.*

Proof. We proceed by induction on i. The beginning of the induction is easy, so suppose the result true for $i-1$.

The map $X[i-1] \times_{Y[i-1]} Y[i] \to Y[i]$ is the pullback of the fibration $X[i-1] \to Y[i-1]$, hence a fibration. It follows that the composite $X[i] \to X[i-1] \times_{Y[i-1]} Y[i] \to Y[i]$ is a fibration. $\square$

Lemma 1.5. *Suppose $X[*] \to Y[*]$ is a levelwise map of tow-G-spectra and let $P[i] = X[i-1] \times_{Y[i-1]} Y[i]$. If $X[*] \to Y[*]$ is a fibration and a weak equivalence, then $X[*] \to P[*]$ is also a weak equivalence.*

Proof. Let $F[i]$ be the fiber of $X[i] \to Y[i]$. From the long exact sequences, it follows that $F[*]$ is acyclic. By the preceding lemma, we have the following diagram of fibrations:

$$\begin{tikzcd}
F[i-1] \arrow[r] \arrow[d, equal] & P[i] \arrow[r] \arrow[d] & Y[i] \arrow[d] \\
F[i-1] \arrow[r] & X[i-1] \arrow[r] & Y[i-1]
\end{tikzcd}$$

Hence, the fiber of $P[*] \to Y[*]$ is also $F[*]$, hence $P[*] \to Y[*]$ is a weak equivalence. It follows that so is $X[*] \to P[*]$. $\square$

Lemma 1.6. *If $X[*] \to Y[*]$ is a fibration, $B[*] \to Y[*]$ is a levelwise map, and $A[*] \to B[*]$ is the pullback, then $A[*] \to B[*]$ is a fibration.*

Proof. It suffices to show that, for each i, $A[i]$ is the pullback in the following diagram:

$$\begin{tikzcd}
A[i] \arrow[r] \arrow[d] & X[i] \arrow[d] \\
A[i-1] \times_{B[i-1]} B[i] \arrow[r] & X[i-1] \times_{Y[i-1]} Y[i]
\end{tikzcd}$$

To see that $A[i]$ is the pullback, consider the following diagram:

$$\begin{tikzcd}
 & X[i] \arrow[rr, equal] \arrow[dd] & & X[i] \arrow[dd] \\
Q \arrow[ur] \arrow[rr] \arrow[dd] & & A[i] \arrow[ur] \arrow[dd] & \\
 & X[i-1] \times_{Y[i-1]} Y[i] \arrow[rr] \arrow[dd] & & Y[i] \arrow[dd] \\
A[i-1] \times_{B[i-1]} B[i] \arrow[ur] \arrow[rr] \arrow[dd] & & B[i] \arrow[ur] \arrow[dd] & \\
 & X[i-1] \arrow[rr] & & Y[i-1] \\
A[i-1] \arrow[ur] \arrow[rr] & & B[i-1] \arrow[ur] &
\end{tikzcd}$$

In this diagram, the bottom face, the lower front and back faces, and the top left and right faces are pullback squares; Q is the pullback in question. Chasing the diagram shows that, given compatible maps from an object to $X[i]$ and $B[i]$, there is a unique map into Q making the diagram commute. It follows that Q is the pullback $B[i] \times_{Y[i]} X[i]$, that is, $Q \cong A[i]$ as desired. $\square$

Corollary 1.7. *If $X[*] \to Y[*]$ is a fibration and $F[*]$ is the* (*levelwise*) *fiber, then $F[*]$ is fibrant.* $\square$

Now we turn to the question of approximating maps with fibrations. The basic construction is simple, based on iterating the usual construction for spaces or spectra. Recall that, if $f : A \to B$ is a map of G-spectra, we write

$Ff = A \times_B B^I$ and get a factorization $A \to Ff \to B$ where the first map is a homotopy equivalence and the second is a Hurewicz fibration.

Definition 1.8. Let $f : A[*] \to B[*]$ be a levelwise map of tow-G-spectra. We define a factorization of f as

$$A[*] \xrightarrow{\phi} Ff[*] \xrightarrow{f'} B[*]$$

as follows. Adopting the convention that $X[-1] = *$ for every tower, suppose that $Ff[i-1]$ has been constructed. Define $Ff[i]$ in the usual way so that

$$A[i] \xrightarrow{\phi} Ff[i] \longrightarrow Ff[i-1] \times_{B[i-1]} B[i]$$

is a factorization of $A[i] \to Ff[i-1] \times_{B[i-1]} B[i]$ into a homotopy equivalence followed by a Hurewicz fibration. When $B[*] = *$, we write $FA[*]$ for $Ff[*]$, so that $FA[*]$ is a fibrant approximation of $A[*]$.

It is straightforward to show that F, defined on levelwise arrows, is functorial on pro-maps between such arrows. In particular, $FA[*]$ is functorial on the category of tow-G-spectra and pro-maps. It is clear that ϕ is a natural (levelwise) weak equivalence, and that f' is a fibration. We wish to show that, when $A[*]$ and $B[*]$ are fibrant, then ϕ is a pro-homotopy equivalence. To that end, we need the following idea.

Definition 1.9. Let B and C be G-spectra. Two maps $\phi : B \to C$ and $\psi : C \to B$ are *adjoint homotopy equivalences* if we are given homotopies $\eta : 1_B \to \psi\phi$ (the *unit*) and $\varepsilon : \phi\psi \to 1_C$ (the *counit*) such that the composites

$$\phi \xrightarrow{\phi\eta} \phi\psi\phi \xrightarrow{\varepsilon\phi} \phi$$

and

$$\psi \xrightarrow{\eta\psi} \psi\phi\psi \xrightarrow{\psi\varepsilon} \psi$$

are each homotopic to the constant homotopy, rel endpoints.

The point of this definition is that it allows us to lift homotopy equivalences as in the following result.

Proposition 1.10. *Suppose that, in the following diagram, ϕ and ψ are adjoint homotopy equivalences with unit and counit η and ε. Suppose also that α and β are fibrations and that f' is the usual approximation of f by a fibration and $\tilde{\phi}$ is the usual inclusion. Then $\tilde{\psi}$ exists making the diagram commute and $\tilde{\phi}$ and $\tilde{\psi}$ are adjoint homotopy equivalences with unit and counit $\tilde{\eta}$ and $\tilde{\varepsilon}$ covering η and ε:*

$$\begin{tikzcd}
A \arrow[r, "\tilde{\phi}"] \arrow[dr, "f"'] \arrow[dd, "\alpha"'] & Ff \arrow[r, "\tilde{\psi}"] \arrow[d, "f'"] & A \arrow[dd] \\
 & D \arrow[d, "\beta"] & \\
B \arrow[r, "\phi"'] & C \arrow[r, "\psi"'] & B
\end{tikzcd}$$

Moreover, this construction is natural, in the following sense. Suppose we have a commutative diagram as follows:

$$\begin{tikzcd}
& A' \arrow[rr, "f'"] \arrow[dd, "\alpha'"] & & D' \arrow[dd, "\beta'"] & & \\
A \arrow[ur] \arrow[rr] \arrow[dd, "\alpha"] & & D \arrow[ur] \arrow[dd, "\beta"] & & & \\
& B' \arrow[rr, "\phi'"] & & C' \arrow[rr, "\psi'"] & & B' \\
B \arrow[ur] \arrow[rr, "\phi"'] & & C \arrow[ur] \arrow[rr, "\psi"'] & & B \arrow[ur] &
\end{tikzcd}$$

Suppose that α' is a fibration compatible with α in the sense that there exist lifting functions for each, compatible in the obvious way. Suppose the same for β' and β. Suppose also that ϕ' and ψ' are adjoint homotopy equivalences with unit and counit and homotopies exhibiting them as such compatible with those of ϕ and ψ. Then, the maps $\tilde{\psi}$ and $\tilde{\psi}'$ constructed, and the units, counits, and homotopies constructed along with them, are all compatible.

Proof. Let us begin by establishing some notation for dealing with these various maps. Recall that $Ff = A \times_D D^I$. We write

$$\tilde{\phi}(a) = \big(a, \kappa\big(f(a)\big)\big)$$

where $\kappa(d)$ denotes the constant path at d. Technically, what this really means is that $\tilde{\phi}$ is the map into the pullback determined by the identity map $A \to A$ and the evident composite $A \to D \to D^I$, but we find the functional notation convenient.

To describe $\tilde{\psi}$ we need to use a lifting function λ for the fibration α. That is, writing $A \times_B B^I$ for the pullback in the following square, $\lambda : A \times_B B^I \to A^I$ is a section of the evident map $A^I \to A \times_B B^I$:

$$\begin{tikzcd}
A \times_B B^I \arrow[r] \arrow[d] & B^I \arrow[d, "p_0"] \\
A \arrow[r, "\alpha"'] & B
\end{tikzcd}$$

We can then write

$$\tilde{\psi}(a, \omega) = p_1\lambda\big(a, \eta\big(\alpha(a), -\big) * \psi\beta\omega\big).$$

Here, ω is a path in D, $\eta(\alpha(a), -)$ is the path determined by η, from $\alpha(a)$ to $\psi\phi\alpha(a)$, and "$*$" denotes path composition.

It is easy to see that the diagram now commutes. More interesting is the construction of the homotopies $\tilde{\eta}$ and $\tilde{\varepsilon}$. We begin with $\tilde{\eta}$. From the formulas above we get

$$\begin{aligned}
\tilde{\psi}\tilde{\phi}(a) &= p_1\lambda\big(a, \eta\big(\alpha(a), -\big) * \psi\beta\kappa\big(f(a)\big)\big) \\
&= p_1\lambda\big(a, \eta\big(\alpha(a), -\big) * \kappa\big(\psi\phi\alpha(a)\big)\big).
\end{aligned}$$

Fig. 1 describes the construction of $\tilde{\eta}$. The figure defines a homotopy $A \times I \to A \times_B B^I$ applied to $a \in A$, with the homotopy coordinate running from bottom to top. If the dashed line is at height s, on its first half it follows $\eta(\alpha(a), u)$ for u running from 0 to s; on its second half it remains constant at $\eta(\alpha(a), s)$. We then apply $p_1\lambda$ to get $\tilde{\eta} : A \times I \to A$.

To define $\tilde{\varepsilon}$, we first note that

$$\tilde{\phi}\tilde{\psi}(a, \omega) = \big(p_1\lambda\big(a, \eta\big(\alpha(a), -\big) * \psi\beta\omega\big), \kappa\big(fp_1\lambda\big(a, \eta\big(\alpha(a), -\big) * \psi\beta\omega\big)\big)\big).$$

If we write $d = fp_1\lambda(a, \eta(\alpha(a), -) * \psi\beta\omega)$, then note that

$$\begin{aligned}
\beta(d) &= \phi\alpha p_1\lambda\big(a, \eta\big(\alpha(a), -\big) * \psi\beta\omega\big) \\
&= \phi p_1\big(\eta\big(\alpha(a), -\big) * \psi\beta\omega\big) \\
&= \phi\psi\beta\omega(1).
\end{aligned}$$

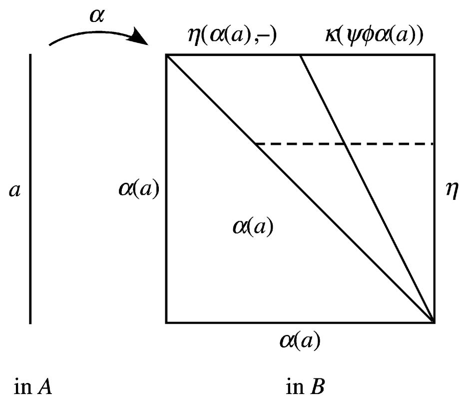

Fig. 1. $\tilde{\eta}$.

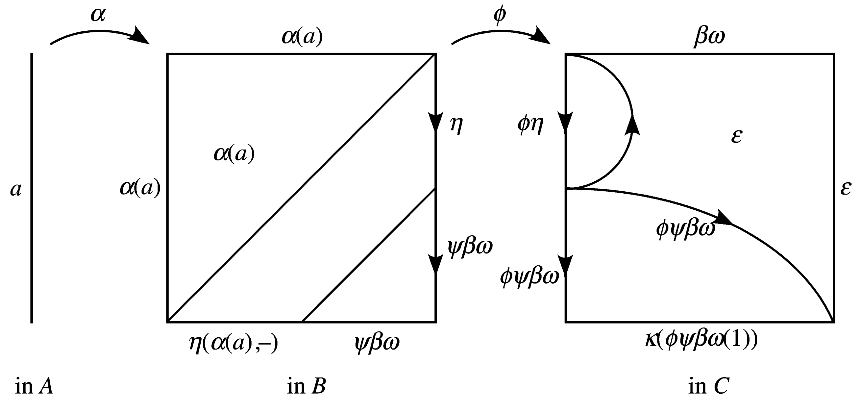

Fig. 2. $\tilde{\varepsilon}$.

Fig. 2 describes the construction of $\tilde{\varepsilon}$. The figure describes a homotopy $Ff \times I \to A \times_B B^I \times_C C^I$, applied to $(a, \omega) \in Ff$, with the homotopy coordinate again running bottom to top. In the square on the right, the large region labeled ε is filled in with the homotopy $\varepsilon : \phi\psi\beta\omega \to \beta\omega$. The unlabeled semicircular path is, from bottom to top, a copy of the path

$$\varepsilon\big(\beta\omega(0), -\big) : \phi\psi\beta\omega(0) \to \beta\omega(0).$$

Now, $\beta\omega(0) = \beta f(a) = \phi\alpha(a)$, so this path is also

$$\varepsilon\big(\phi\alpha(a), -\big) : \phi\psi\phi\alpha(a) \to \phi\alpha(a).$$

If we follow the path labeled $\phi\eta$ and then the semicircular path, we are following the homotopy $\phi\eta * \varepsilon\phi$ applied to the point $\alpha(a)$. By the assumption that we have an adjoint homotopy equivalence, this path is null homotopic. Hence, we can fill in the remaining half disc they surround.

To obtain $\tilde{\varepsilon}$ we now apply $fp_1\lambda$ to the homotopy into $A \times_B B^I$. Combining with the homotopy into C^I we get a homotopy into $D \times_C C^I$. Using the evident lifts on the bottom and top of the square, we now lift to a homotopy in D. Combined with the endpoint of the lift to A, we now have the homotopy $\tilde{\varepsilon} : Ff \times I \to Ff$, from $\tilde{\phi}\tilde{\psi}$ to the identity, covering ε, as desired.

Fig. 3 shows, from bottom to top, the homotopy

$$\tilde{\phi} \xrightarrow{\tilde{\phi}\tilde{\eta}} \tilde{\phi}\tilde{\psi}\tilde{\phi} \xrightarrow{\tilde{\varepsilon}\tilde{\phi}} \tilde{\phi}.$$

The square in the bottom right of the figure has each horizontal line a constant path. Now, by sliding the contents of these boxes off the right edge, and sliding in a constant map from the left, we see that this homotopy is homotopic, rel endpoints, to a constant homotopy.

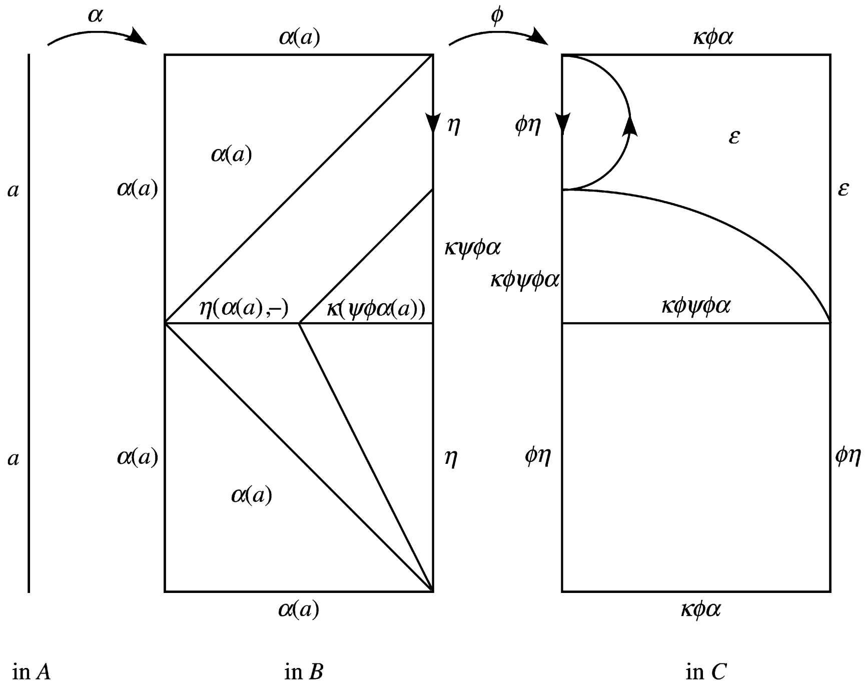

Fig. 3. $(\tilde{\varepsilon}\tilde{\phi}) \circ (\tilde{\phi}\tilde{\eta})$.

Fig. 4 shows, from bottom to top, the homotopy

$$\tilde{\psi} \xrightarrow{\tilde{\eta}\tilde{\psi}} \tilde{\psi}\tilde{\phi}\tilde{\psi} \xrightarrow{\tilde{\psi}\tilde{\varepsilon}} \tilde{\psi}$$

applied to a point (a, ω). Precisely, we get the homotopy by lifting this picture to A and then taking the right edge. Tracing through the definitions, we can see that the bottom edge is lifted by applying λ to $\eta * \psi\beta\omega$ and then following by a constant path, while the top edge is lifted to a constant path followed by λ applied to $\eta * \psi\beta\omega$. Now, Fig. 4 is homotopic, rel top, bottom, and left sides, to Fig. 5. Such a homotopy gives a map of a cube into B with Figs. 4 and 5 as the bottom and top faces. We can lift five of the faces of this cube into A. On the top face we use the lift that is constant on the triangles and always lifts $\eta * \psi\beta\omega$ to $\lambda(\eta * \psi\beta\omega)$. We then use the fibration property to lift the whole cube. The right side of the cube then gives the desired homotopy.

For the last statement of the proposition, the constructions above, applied to A', B', C', and D', will be compatible, given all of the compatibility assumed in the hypotheses. $\square$

As a corollary we get the following result.

Theorem 1.11. *If $X[*]$ and $Y[*]$ are fibrant and $f : X[*] \to Y[*]$ is a levelwise map, then we can factor f as*

$$X[*] \xrightarrow{\phi} Ff[*] \xrightarrow{f'} Y[*]$$

where ϕ is a pro-homotopy equivalence and f' is a fibration. Moreover, suppose we have the following levelwise diagram:

$$\begin{tikzcd} X[*] \arrow[r, "f"] \arrow[d] & Y[*] \arrow[d] \\ X'[*] \arrow[r, "f'"'] & Y'[*] \end{tikzcd}$$

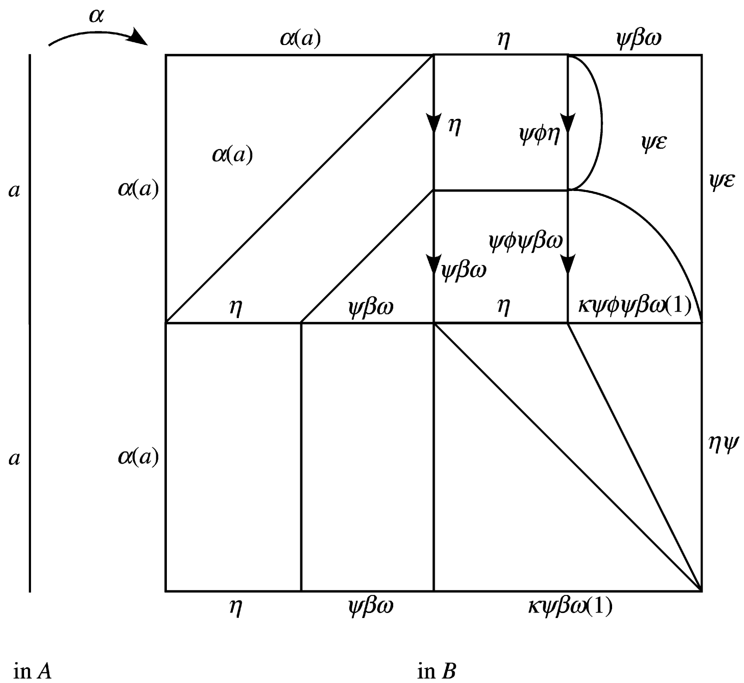

Fig. 4. $(\tilde{\psi}\tilde{\varepsilon}) \circ (\tilde{\eta}\tilde{\psi})$.

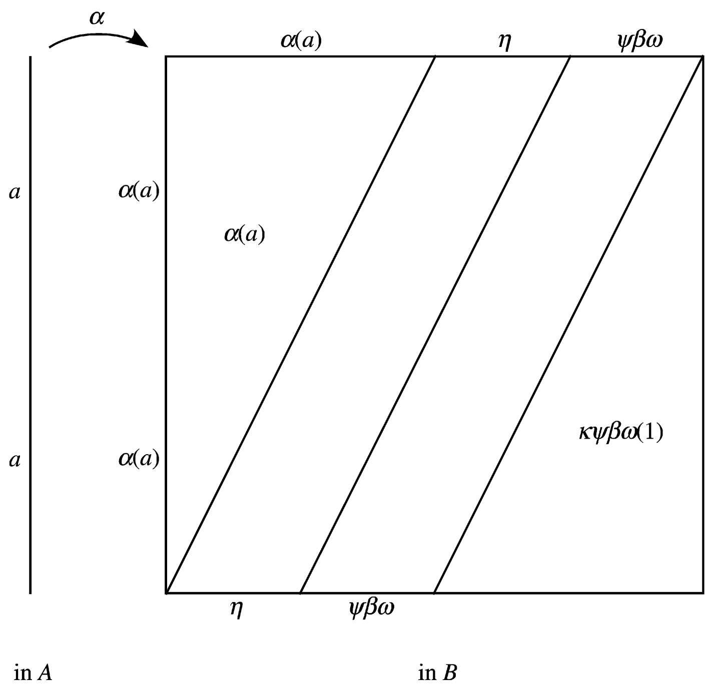

Fig. 5.

If the maps $X[*] \to X'[*]$ and $Y[*] \to Y'[*]$ preserve (*levelwise*) *lifting functions, then the homotopy equivalence is natural, in the sense that the arrows $X[*] \to X'[*]$ and $Ff[*] \to Ff'[*]$ are homotopy equivalent by the homotopy inverses and homotopies constructed as in Proposition* 1.10*.*

Proof. We have already constructed $Ff[*]$ and the maps ϕ and f', and we know that f' is a fibration. We construct a pro-homotopy inverse ψ to ϕ inductively. Suppose that we have already shown that the map $\phi : X[i-1] \to Ff[i-1]$ is an adjoint homotopy equivalence with inverse equivalence ψ. Consider the following diagram:

$$\begin{tikzcd}
X[i] \arrow[r, "\phi"] \arrow[dd] \arrow[dr] & Ff[i] \arrow[r, dashed, "\psi"] \arrow[d] & X[i] \arrow[dd] \\
 & Ff[i-1] \times_{Y[i-1]} Y[i] \arrow[d] & \\
X[i-1] \arrow[r, "\phi"'] & Ff[i-1] \arrow[r, "\psi"'] & X[i-1]
\end{tikzcd}$$

We fill in the top right map as in Proposition 1.10, obtaining a map ψ so that ϕ and ψ are adjoint homotopy equivalences, with homotopies compatible with those at level $i-1$. By this compatibility, $\psi : Ff[*] \to X[*]$ is a pro-homotopy inverse to ϕ.

The naturality of this homotopy equivalence follows from the naturality of Proposition 1.10. $\square$

The following lemma will be used in proving that, up to homotopy, $FA[*]$ is the "free" fibrant tow-G-spectrum on $A[*]$.

Lemma 1.12. *Consider the following diagram of spectra*:

$$\begin{tikzcd}
A \arrow[r, "\phi"] \arrow[dr, "f"'] & Ff \arrow[r, "\tilde{\phi}, \tilde{\psi}"] \arrow[d, "\beta"] & F(\phi\beta) \arrow[d] \\
 & B \arrow[r, "\phi \simeq \psi"'] & C
\end{tikzcd}$$

Suppose given a homotopy $h : \phi \to \psi$ and suppose further that $\phi f = \psi f$, hf is constant, and h is constant on some initial segment $[0, s]$, where $0 < s \leqslant 1$. Let $\tilde{\phi}$ and $\tilde{\psi}$ be the lifts of ϕ and ψ defined by

$$\tilde{\phi}(a, \omega) = \big(a, \omega, \kappa\phi\omega(1)\big)$$

and

$$\tilde{\psi}(a, \omega) = \big(a, \kappa f(a), \psi\omega\big).$$

Then, there exists a homotopy $\tilde{h} : \tilde{\phi} \to \tilde{\psi}$ covering h, constant on some initial segment, and such that $\tilde{h}\phi$ is constant. Moreover, this construction can be made natural in the given data.

Proof. The homotopy is the one described in Fig. 6. Here, the block in C above the line at height s is the restriction $h|[s, 1]$ applied to ω. It is straightforward to check that the homotopy $\tilde{h}$ described by this picture satisfies the requirements of the lemma, including naturality. $\square$

As a corollary we get the following.

Proposition 1.13. *If $A[*]$ is any tow-G-spectrum, then $\phi \simeq F\phi : FA[*] \to F^2A[*]$. Moreover, this homotopy can be constructed naturally in $A[*]$.*

Proof. We construct the homotopy by induction. The inductive step uses the following diagram:

$$\begin{tikzcd}
A[i] \arrow[r, "\phi"] \arrow[d] & FA[i] \arrow[r, "\phi, F\phi"] \arrow[d] & F^2A[i] \arrow[d] \\
A[i-1] \arrow[r, "\phi"'] & FA[i-1] \arrow[r, "\phi \simeq F\phi"'] & F^2A[i-1]
\end{tikzcd}$$

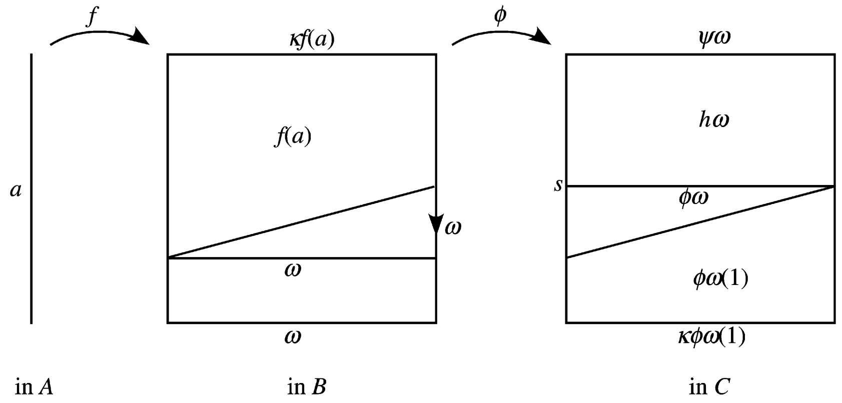

Fig. 6.

In the trivial case $i = 0$ we can take the homotopy to be constant. So, we can assume, in general, that the homotopy at level $i - 1$ satisfies the hypotheses of the preceding lemma, and we can construct a homotopy at level i that does also. Naturality follows from the naturality of the preceding lemma. □

Theorem 1.14. *The fibrant approximation functor F is left adjoint to the inclusion of the category of fibrant tow-G-spectra in the category of all tow-G-spectra, when we pass to homotopy. That is, writing $[-,-]_G^h$ for the set of maps between tow-G-spectra, modulo pro-homotopy,*

$$\big[A[*], X[*]\big]_G^h \cong \big[FA[*], X[*]\big]_G^h$$

for any $A[*]$, if $X[*]$ is fibrant.

Proof. The unit of the adjunction is given by $\phi: A[*] \to FA[*]$. By Theorem 1.11, ϕ is a pro-homotopy equivalence when applied to a fibrant tow-G-spectrum. We let $\psi: FX[*] \to X[*]$ be the inverse isomorphism in the homotopy category, when $X[*]$ is fibrant; this is the counit. To show that we have an adjunction it then suffices to show that the following two composites are each the identity:

$$X[*] \xrightarrow{\phi} FX[*] \xrightarrow{\psi} X[*]$$

for fibrant $X[*]$, and

$$FA[*] \xrightarrow{F\phi} F^2A[*] \xrightarrow{\psi} FA[*]$$

for any $A[*]$. The first composite is the identity by definition of ψ. For the second composite, we showed in Proposition 1.13 that $F\phi \simeq \phi$, so this composite is also the identity. □

2. Sequences of towers

As mentioned in the introduction, we work in a certain full subcategory of ind-pro-$G\mathscr{S}$:

Definition 2.1. The category seq-tow-$G\mathscr{S}$ is the full subcategory of ind-pro-$G\mathscr{S}$ on the objects isomorphic to functors on $\mathbb{N} \times \mathbb{N}^{\mathrm{op}}$, where $\mathbb{N}$ denotes the category of non-negative integers with arrows going from small integers to larger ones. The notion of homotopy in seq-tow-$G\mathscr{S}$ is the same as in ind-pro-$G\mathscr{S}$; a homotopy of two ind-pro-maps out of an ind-pro-G-spectrum X is an ind-pro-map out of $X \wedge I_+$, the objectwise smash product.

If X is a seq-tow-G-spectrum, we shall write $X[i,j]$ for the spectrum at $(i,j) \in \mathbb{N} \times \mathbb{N}^{\mathrm{op}}$, or $X[*,*]$ for X. Thus, the first index is the ind-index and the second the pro-index.

Our seq-tow-G-spectra are represented by objects with very strict internal maps, but morphisms between them are allowed to be arbitrary ind-pro-G-maps. However, it turns out that we can make individual maps strict as well:

Proposition 2.2. *If $f : X[*, *] \to Y[*, *]$ is a map of seq-tow-G-spectra, then f is isomorphic to a levelwise map, that is, a natural transformation of functors on $\mathbb{N} \times \mathbb{N}^{\mathrm{op}}$.*

Proof. We may assume that X and Y are functors on $\mathbb{N} \times \mathbb{N}^{\mathrm{op}}$ and, as for any ind-map, we may assume that f is levelwise with respect to the ind-index. Define a seq-tow-G-spectrum Z by $Z[i, j] = Y[i, j]$ if $i \leqslant j$ and $Z[i, j] = *$ if $i > j$. The obvious levelwise map $Y \to Z$ is an ind-pro-isomorphism, so we may assume that $Y[i, j] = *$ for $i > j$.

We now construct, by induction, integers $k_1 < k_2 < \cdots$ and maps $X[i, k_j] \to Y[i, j]$ representing f as a natural transformation. Assuming we have done so for integers less than j, choose k_j large enough that we have representatives $X[i, k_j] \to Y[i, j]$ of f for $i \leqslant j$, compatible with the choices made for smaller j; we can do this because there are only finitely many conditions. For $i > j$, we simply take the unique map $X[i, k_j] \to Y[i, j] = *$.

Since these maps represent the original f, if we let $W[i, j] = X[i, k_j]$, the resulting levelwise map $W[*, *] \to Y[*, *]$ is isomorphic to f. $\square$

By naturality, several of the results of the preceding section extend to seq-tow-G-spectra. We first introduce the following definition.

Definition 2.3. A levelwise map $X[*, *] \to Y[*, *]$ of seq-tow-G-spectra is a *fibration* if $X[i, *] \to Y[i, *]$ is a fibration for all i, with the lifting functions at level i compatible with those at level $i + 1$. A seq-tow-G-spectrum $X[*, *]$ is *fibrant* if the map $X[*, *] \to *$ is a fibration; equivalently, each $X[i, j] \to X[i, j - 1]$ is a fibration, with lifting function compatible with that for $X[i + 1, j] \to X[i + 1, j - 1]$.

Theorem 2.4. *If $X[*, *]$ and $Y[*, *]$ are fibrant and $f : X[*, *] \to Y[*, *]$ is a levelwise map, then we can factor f as*

$$X[*, *] \xrightarrow{\phi} Ff[*, *] \xrightarrow{f'} Y[*, *]$$

where ϕ is a seq-tow-G-homotopy equivalence and f' is a fibration.

Proof. This follows by applying Theorem 1.11 to each seq-level and appealing to the naturality of that result. $\square$

Theorem 2.5. *The fibrant approximation functor F is left adjoint to the inclusion of the category of fibrant seq-tow-G-spectra in the category of all seq-tow-G-spectra, when we pass to homotopy. That is, writing $[-, -]^h_G$ for the set of maps between seq-tow-G-spectra, modulo ind-pro-homotopy,*

$$\bigl[A[*, *], X[*, *]\bigr]^h_G \cong \bigl[FA[*, *], X[*, *]\bigr]^h_G$$

for any $A[*, *]$, if $X[*, *]$ is fibrant.

Proof. The proof is the same as for Theorem 1.14, using the naturality of Proposition 1.13. $\square$

3. Cell complexes

Definition 3.1. An *n-sphere object* is a tow-G-spectrum $S[*]$ such that each $S[i]$ is a wedge of n-spheres, by which we mean spectra of the form $G/H_+ \wedge S^n$. Disc objects are defined similarly.

For each n-sphere object $S[*]$ there is a corresponding $(n + 1)$-disc object $D[*]$ obtained by filling in the spheres and a map $S[*] \to D[*]$ that we call an *n-cell object*.

Our cell complexes will be seq-tow-G-spectra. We shall tacitly consider any tow-G-spectrum to be a seq-tow-G-spectrum with a single seq-level.

Definition 3.2. A *relative cell complex* is a seq-tow-map $A[*, *] \to B[*, *]$ of seq-tow-G-spectra where $B[*, *] = \mathrm{colim}_n\, A_n[*, *]$, $A[*, *] = A_0[*, *]$, and each $A_{n+1}[*, *]$ is obtained from $A_n[*, *]$ by pushing out along a cell object. We call $\{A_n[*, *]\}$ the *attachment sequence*. If $A[*, *] = *$, we call $B[*, *]$ a *cell complex*.

Remark 3.3. In the colimit $\mathrm{colim}_n\, A_n[*,*]$ appearing in the definition above, the maps are levelwise. In this case, we can take as the colimit the seq-tow-G-spectrum $C[i,j] = A_i[i,j]$. It is easy to check that this is the categorical colimit.

Theorem 3.4. *If $f : A[*,*] \to B[*,*]$ is a seq-tow-map, then there exists a relative cell complex $A[*,*] \to \Gamma f[*,*]$ and a seq-tow-map $\xi : \Gamma f[*,*] \to B[*,*]$ such that*

(1) *$\Gamma f[i,*] \to B[i,*]$ is levelwise for $i \geqslant 1$, and*
(2) *for each pair of integers $m \leqslant n$, there exists a cofinal sequence of indices i such that $\bar\pi_p(\xi) : \bar\pi_p \Gamma f[i,j] \cong \bar\pi_p B[i,j]$ for all j and all $m \leqslant p \leqslant n$.*

Proof. We can assume that f is a levelwise map. Let $A_0[*,*] = A[*,*]$. We construct a sequence of seq-tow-G-spectra $\{A_k[*,*]\}$ inductively, with levelwise maps $A_k[*,*] \to B[*,*]$ such that $\bar\pi_p A_k[\ell,j] \to \bar\pi_p B[\ell,j]$ is an isomorphism for $-\ell \leqslant p \leqslant \ell$ for all $\ell \leqslant k$ and for all j. Assume $A_{k-1}[*,*]$ already constructed.

If we want to kill the kernel of the map $\bar\pi_p A_{k-1}[k,j] \to \bar\pi_p B[k,j]$, we should attach all the cells coming from diagrams of the form

$$\begin{tikzcd} G/H_+ \wedge S^p \arrow[r] \arrow[d] & A_{k-1}[k,j] \arrow[d] \\ G/H_+ \wedge D^{p+1} \arrow[r] & B[k,j] \end{tikzcd}$$

Let $S[j] \to D[j]$ be the wedge of all the maps in the diagram above. $S[*]$ and $D[*]$ are tow-G-spectra by composition and we get the following diagram:

$$\begin{tikzcd} S[*] \arrow[r] \arrow[d] & A_{k-1}[k,*] \arrow[d] \\ D[*] \arrow[r] & B[k,*] \end{tikzcd}$$

Now, attach $(p+1)$-cells in this way for $-k \leqslant p \leqslant k+1$ and let $A_k[*,*]$ be the result of this finite number of attachments. It is not difficult to see that $A_k[*,*]$ satisfies the inductive hypothesis.

Now let $\Gamma f[*,*] = \mathrm{colim}_k\, A_k[*,*]$. As we remarked above, we can identify $\Gamma f[*,*]$ as the seq-tow-G-spectrum given by $\Gamma f[k,j] = A_k[k,j]$. The claims in the theorem now follow from the construction of $A_k[*,*]$. $\square$

4. Weak equivalences

The object of this section is to show that, for certain kinds of seq-tow-G-spectra, weak equivalences are homotopy equivalences.

Lemma 4.1. *If $F[*]$ is fibrant with $\bar\pi_n F[*] = 0$ and $\bar\pi_{n+1} F[*] = 0$, and $\alpha : S[*] \to F[*]$ is a pro-map from an n-sphere object, then α extends to $\beta : D[*] \to F[*]$ where $D[*]$ is the corresponding disc object.*

Proof. We construct, by induction, compatible extensions $\beta_i : D[j_i] \to F[i]$. We begin with the case $i = 0$. We first choose a k_0 such that $\bar\pi_{n+1} F[k_0] \to \bar\pi_{n+1} F[0]$ is 0. Now, using the fact that $\bar\pi_n F[*]$ is 0, we choose an extension $\bar\beta_0 : D[j_0] \to F[k_0]$ of the map from $S[j_0]$, for some sufficiently large j_0. We let β_0 be the composite $D[j_0] \to F[k_0] \to F[0]$.

Now suppose that we have constructed β_i as a composite of $\bar\beta_i : D[j_i] \to F[k_i]$ with the structure map $F[k_i] \to F[i]$, where k_i is such that the map $\bar\pi_{n+1} F[k_i] \to \bar\pi_{n+1} F[i]$ is 0. Choose a $k_{i+1} \geqslant k_i$ such that $\bar\pi_{n+1} F[k_{i+1}] \to \bar\pi_{n+1} F[i+1]$ is 0. Again using the fact that $\bar\pi_n F[*]$ is 0, we can choose a $j_{i+1} \geqslant j_i$ and a map $\bar\beta_{i+1} : D[j_{i+1}] \to F[k_{i+1}]$ extending the map from $S[j_{i+1}]$. We now compare the composite

$$D[j_{i+1}] \xrightarrow{\bar\beta_{i+1}} F[k_{i+1}] \longrightarrow F[k_i] \longrightarrow F[i]$$

to the composite

$$D[j_{i+1}] \longrightarrow D[j_i] \xrightarrow{\bar\beta_i} F[k_i] \longrightarrow F[i].$$

These two maps agree on $S[j_{i+1}]$ hence glue together to define an element of $\bar\pi_{n+1}F[i]$. However, since these maps factor through $F[k_i]$, this homotopy element is 0. Therefore, the two composites are homotopic rel $S[j_{i+1}]$. Using the assumption that $F[k_{i+1}] \to F[i]$ is a fibration, we can adjust $\bar\beta_{i+1}$, rel boundary, so that the two composites are in fact equal. This completes the inductive step. $\square$

Proposition 4.2. *Let $f : X[*] \to Y[*]$ be a fibration.*

(1) *If $\bar\pi_n f$, $\bar\pi_{n+1} f$, and $\bar\pi_{n+2} f$ are all isomorphisms, then, for every n-sphere object $S[*]$ and corresponding disc object $D[*]$, we can always find a lift in the following diagram*:

$$\begin{tikzcd} S[*] \arrow[r] \arrow[d] & X[*] \arrow[d] \\ D[*] \arrow[r] \arrow[ur, dashed] & Y[*] \end{tikzcd}$$

(2) *If we can always find a lift in the diagram above, for every n-sphere object $S[*]$ and corresponding disc object $D[*]$, and for every $(n+1)$-sphere object $S[*]$ and corresponding disc object $D[*]$, then $\bar\pi_n f$ is an isomorphism.*
(3) *The map f is a weak equivalence if and only if, for every sphere object $S[*]$ and corresponding disc object $D[*]$, we can always find a lift in the diagram above.*

Proof. For the first statement, suppose that $\bar\pi_n f$, $\bar\pi_{n+1} f$, and $\bar\pi_{n+2} f$ are all isomorphisms, and consider the following lifting problem:

$$\begin{tikzcd} S[*] \arrow[r, "\alpha"] \arrow[d] & X[*] \arrow[d] \\ D[*] \arrow[r, "\gamma"] \arrow[ur, dashed, "\beta"] & Y[*] \end{tikzcd}$$

By Lemma 1.3, we can find a lift in the following diagram, where the map $S[*] \wedge I_+ \to D[*]$ is a contraction to the basepoint:

$$\begin{tikzcd} S[*] \arrow[rr, "\alpha"] \arrow[d, "i_0"'] & & X[*] \arrow[d] \\ S[*] \wedge I_+ \arrow[r] \arrow[urr, dashed, "h"] & D[*] \arrow[r, "\gamma"] & Y[*] \end{tikzcd}$$

Note that h_1 maps $S[*]$ into the fiber $F[*]$ and that $F[*]$ is fibrant and, by the long exact homotopy sequence, satisfies the hypotheses of Lemma 4.1. So, we can extend h_1 to a map $\bar\beta : D[*] \to F[*]$. Now consider the following lifting problem:

$$\begin{tikzcd} S[*] \wedge I_+ \cup D[*] \arrow[r, "h\cup\bar\beta"] \arrow[d, "i_1"'] & X[*] \arrow[d] \\ D[*] \wedge I_+ \arrow[r, "c"'] \arrow[ur, dashed, "H"] & Y[*] \end{tikzcd}$$

Here, c is a contraction along γ extending the contraction of $S[*]$ we used earlier. That we can find the lift H follows from Lemma 1.3. The map $\beta = H_0$ is then the lift we were seeking.

For the second statement, suppose that we can find the indicated lifts. Since the category of pro-Mackey functors is abelian [1], it suffices to show that the kernel and cokernel of $\bar{\pi}_n X[*] \to \bar{\pi}_n Y[*]$ are pro-zero. We first show that the kernel is pro-zero. For each i, consider all possible diagrams of spectra of the following form:

$$\begin{tikzcd} G/H_+ \wedge S^n \arrow[r] \arrow[d] & X[i] \arrow[d] \\ G/H_+ \wedge D^{n+1} \arrow[r] & Y[i] \end{tikzcd}$$

Let $S[i] \to D[i]$ be the wedge of all of the maps $G/H_+ \wedge S^n \to G/H_+ \wedge D^{n+1}$ used above, and let

$$\begin{tikzcd} S[i] \arrow[r] \arrow[d] & X[i] \arrow[d] \\ D[i] \arrow[r] & Y[i] \end{tikzcd}$$

be obtained by taking the wedge of all the maps in the diagrams above. By composition, $S[*]$ and $D[*]$ are tow-G-spectra and we have the following diagram, in which we can find a lift β by assumption:

$$\begin{tikzcd} S[*] \arrow[r] \arrow[d] & X[*] \arrow[d] \\ D[*] \arrow[r] \arrow[ur, dashed, "\beta"] & Y[*] \end{tikzcd}$$

To say that the kernel of $\bar{\pi}_n X[*] \to \bar{\pi}_n Y[*]$ is pro-zero is to say that, for each j there is an i such that

$$\ker\bigl(\bar{\pi}_n X[i] \to \bar{\pi}_n Y[i]\bigr) \to \bar{\pi}_n X[j]$$

is the zero map. That this is true follow from the existence of the lift β above.

To show that the cokernel is zero, we need to show that, for each j, there exists an i such that

$$\operatorname{im}\bigl(\bar{\pi}_n Y[i] \to \bar{\pi}_n Y[j]\bigr) \subset \operatorname{im}\bigl(\bar{\pi}_n X[j] \to \bar{\pi}_n Y[j]\bigr).$$

To show that this is so, it suffices to solve the following lifting problem:

$$\begin{tikzcd} D_-[*] \arrow[r] \arrow[d] & X[*] \arrow[d] \\ S[*] \arrow[r] \arrow[ur, dashed] & Y[*] \end{tikzcd}$$

Here, $D_-[*] \to S[*]$ is formed by taking, at each level, the wedge of all diagrams of the form:

$$\begin{tikzcd} G/H_+ \wedge D^n_- \arrow[r] \arrow[d] & X[i] \arrow[d] \\ G/H_+ \wedge S^n \arrow[r] & Y[i] \end{tikzcd}$$

where D^n_- is one hemisphere of S^n. If $E[i]$ denotes the equator of $S[i]$ and $D_+[i]$ the other hemisphere, we have the following diagram, in which the square on the left is a pushout, $E[*]$ is a sphere object, and $D_+[*]$ is its corresponding disc object:

$$\begin{tikzcd} E[*] \arrow[r] \arrow[d] & D_-[*] \arrow[r] \arrow[d] & X[*] \arrow[d] \\ D_+[*] \arrow[r] \arrow[urr, dashed] & S[*] \arrow[r] & Y[*] \end{tikzcd}$$

By assumption, we can find the indicated lift. Since $S[*]$ is the pushout it follows that we can find the lift we want.

The final statement follows immediately from the first two. $\square$

The following definition may appear a bit arbitrary. However, on towers it reduces to pro-isomorphism of pro-homotopy groups, and was settled on after trying several alternatives, as the one which allows us to get Theorem 4.8 below.

Definition 4.3. A map of seq-tow-G-spectra is a *weak equivalence* if it has a levelwise representative $X[*,*] \to Y[*,*]$ such that, for each pair of integers $m \leqslant n$, there exists a cofinal sequence of indices i such that $\bar{\pi}_p X[i,*] \to \bar{\pi}_p Y[i,*]$ is a pro-isomorphism for all $m \leqslant p \leqslant n$.

Corollary 4.4. *Acyclic fibrations of seq-tow-G-spectra have the right lifting property with respect to relative cell complexes.*

Proof. We construct a lift by induction on the index of the attachment sequence of the cell complex. The inductive step requires a lift in the following diagram, where $S[*] \to D[*]$ is a $(n+1)$-cell object:

$$\begin{tikzcd} S[*] \arrow[r] \arrow[d] & X[*,*] \arrow[d] \\ D[*] \arrow[r] & Y[*,*] \end{tikzcd}$$

We can assume that the cell maps into a seq-index i such that $X[i,*] \to Y[i,*]$ is a $\bar{\pi}_p$ isomorphism for $n \leqslant p \leqslant n+2$. We can then find a lift using the preceding proposition. □

Proposition 4.5. *Weak equivalences between fibrant seq-tow-G-spectra have the homotopy right lifting property with respect to relative cell complexes.*

Proof. Let $f : X[*,*] \to Y[*,*]$ be a weak equivalence of fibrant seq-tow-G-spectra; by Proposition 2.2 we may assume that f is levelwise. By Theorem 2.4, we can factor f as a seq-tow-G-homotopy equivalence $X[*,*] \to Ff[*,*]$ followed by a fibration $Ff[*,*] \to Y[*,*]$. Therefore, consider the following diagram, in which $A[*,*] \to B[*,*]$ is a relative cell complex:

$$\begin{tikzcd} A[*,*] \arrow[r] \arrow[d] & X[*,*] \arrow[r, "\simeq"] \arrow[d] & Ff[*,*] \arrow[dl] \\ B[*,*] \arrow[r] \arrow[urr, dashed] & Y[*,*] & \end{tikzcd}$$

We can find the dotted arrow by the preceding corollary. Composing with an inverse homotopy equivalence $Ff[*,*] \to X[*,*]$ we get a homotopy lifting. □

As a consequence of the preceding proposition, we get the following form of Whitehead's Theorem.

Theorem 4.6. *Let $f : X[*,*] \to Y[*,*]$ be a weak equivalence of fibrant seq-tow-G-spectra. Then, for any cellular seq-tow-G-spectrum $A[*,*]$, we have $f_* : [A,X]^h_G \cong [A,Y]^h_G$.*

Proof. To see that f_* is an epimorphism, apply the preceding proposition to the relative cell complex $* \to A$. To see that it is a monomorphism, apply the proposition to $A \vee A \to A \wedge I_+$. □

Definition 4.7. A *well-behaved* seq-tow-G-spectrum is one that has the form $FA[*,*]$ where $A[*,*]$ is a cellular seq-tow-G-spectrum.

The reason for this name is the following result.

Theorem 4.8. *A weak equivalence between well-behaved seq-tow-G-spectra is a homotopy equivalence.*

Proof. Suppose that $f : FA \to FB$ is a weak equivalence. We have the following chain of isomorphisms:

$$\begin{aligned}[FB, FA]^h_G &\cong [B, FA]^h_G \\ &\cong [B, FB]^h_G \\ &\cong [FB, FB]^h_G.\end{aligned}$$

The first and last of these isomorphisms is given by Theorem 2.5. The middle isomorphism is an application of Whitehead's Theorem. Moreover, it is easy to see that the composite isomorphism is given by composition with f. Let $g : FB \to FA$ be the map corresponding to the identity on FB via this isomorphism. Then fg is homotopic to the identity on FB. Now g is a weak equivalence because f is, so the same argument applied to g produces a map h such that gh is homotopic to the identity. It follows that f and g are inverse homotopy equivalences. □

We now have the machinery in place to invert weak equivalences. If $X[*,*]$ is an arbitrary seq-tow-G-spectrum, we have the following weak equivalences:

$$X[*,*] \leftarrow \Gamma X[*,*] \to F\Gamma X[*,*]$$

where $\Gamma X[*,*]$ is cellular (Theorem 3.4) and $F\Gamma X[*,*]$ is well-behaved. Further, if $f : X[*,*] \to Y[*,*]$ is a seq-tow-map, the functoriality of Γ and F give us the following diagram:

$$\begin{tikzcd} X[*,*] \arrow[d, "f"'] & \Gamma X[*,*] \arrow[l] \arrow[r] \arrow[d, "\Gamma f"'] & F\Gamma X[*,*] \arrow[d, "F\Gamma f"'] \\ Y[*,*] & \Gamma Y[*,*] \arrow[l] \arrow[r] & F\Gamma Y[*,*] \end{tikzcd}$$

If f is a weak equivalence, it is clear from the diagram above that Γf and $F\Gamma f$ are also weak equivalences. By Theorem 4.8, $F\Gamma f$ is then a homotopy equivalence, so becomes an isomorphism on passing to homotopy classes of maps.

Definition 4.9. The *homotopy category* of seq-tow-G-spectra is the one with mapping sets defined by

$$[X, Y]_G = [F\Gamma X, F\Gamma Y]^h_G.$$

By the above discussion, we have the following result.

Theorem 4.10. *The homotopy category of seq-tow-G-spectra is the localization of the category of seq-tow-G-spectra at the class of weak equivalences. That is, any functor on the category of seq-tow-G-spectra that takes weak equivalences to isomorphisms factors uniquely through the homotopy category.*

We can consider the full subcategory $h(\text{tow-}G\mathscr{S})$ of the homotopy category consisting of the tow-G-spectra. In this subcategory the weak equivalences are inverted, but it is not clear that the category is the localization at the weak equivalences. However, we do have the following, which says that we have not inverted too much.

Recall that, if $X[*]$ is a tow-G-spectrum, it defines a tow-Mackey functor-valued $RO(G)$-graded cohomology theory on finite G-CW complexes, by setting

$$\overline{X}^n(A)[*](G/H) = \left[A, \Sigma^n X[*]\right]_H.$$

Let tow-cohom be the category of tow-Mackey functor-valued $RO(G)$-graded cohomology theories on finite G-CW complexes. Let $\kappa :$ tow-$G\mathscr{S} \to$ tow-cohom be the functor that takes pro-spectra to their corresponding cohomology theories.

Definition 4.11. A *pseudo-homotopy category* for tow-$G\mathscr{S}$ is a category $k(\text{tow-}G\mathscr{S})$ and a functor tow-$G\mathscr{S} \to k(\text{tow-}G\mathscr{S})$ such that

(1) every weak equivalence in tow-$G\mathscr{S}$ is mapped to an isomorphism in $k(\text{tow-}G\mathscr{S})$ and
(2) κ factors through $k(\text{tow-}G\mathscr{S})$.

Theorem 4.12. *The category $h(\text{tow-}G\mathscr{S})$ is a pseudo-homotopy category for tow-$G\mathscr{S}$.*

Proof. Consider the following diagram.

$$\begin{tikzcd}
\text{tow-}G\mathscr{S} \arrow[rrrr] \arrow[dr] \arrow[dddrr, bend right, "\kappa"'] & & & & h(\text{tow-}G\mathscr{S}) \arrow[dl] \arrow[dddll, dashed, bend left, "\lambda"] \\
& \text{seq-tow-}G\mathscr{S} \arrow[rr] \arrow[dr, "\kappa'"'] & & h(\text{seq-tow-}G\mathscr{S}) \arrow[dl, "\lambda'"] & \\
& & \text{seq-tow-cohom} & & \\
& & \text{tow-cohom} \arrow[u] & &
\end{tikzcd}$$

Here, seq-tow-cohom is the category of seq-tow-Mackey functor valued $RO(G)$-graded cohomology theories. That λ' exists follows from the fact that $h(\text{seq-tow-}G\mathscr{S})$ is obtained from seq-tow-$G\mathscr{S}$ by inverting weak equivalences. That λ exists then follows from the fact that λ', when restricted to the subcategory $h(\text{tow-}G\mathscr{S})$, takes values in the full subcategory tow-cohom. □

Acknowledgements

We thank Dan Isaksen for some illuminating correspondence. We thank the reviewer for comments that helped us to improve the exposition.

References

[1] M. Artin, B. Mazur, Etale Homotopy, Lecture Notes in Mathematics, vol. 100, Springer, Berlin, 1969, MR MR0245577 (39 #6883).
[2] S.R. Costenoble, S. Waner, A completion theorem for pro-G-spectra, Preprint, 2006.
[3] D.A. Edwards, H.M. Hastings, Čech and Steenrod Homotopy Theories with Applications to Geometric Topology, Lecture Notes in Mathematics, vol. 542, Springer, Berlin, 1976, MR 55 #1347.
[4] H. Fausk, D.C. Isaksen, t-Model structures, math.AT/0511056, 2005.
[5] J.W. Grossman, A homotopy theory of pro-spaces, Trans. Amer. Math. Soc. 201 (1975) 161–176, MR MR0356039 (50 #8511).
[6] D.C. Isaksen, A model structure on the category of pro-simplicial sets, Trans. Amer. Math. Soc. 353 (7) (2001) 2805–2841 (electronic), MR MR2001m:18015.
[7] D.C. Isaksen, Strict model structures for pro-categories, in: Categorical Decomposition Techniques in Algebraic Topology (Isle of Skye, 2001), in: Progr. Math., vol. 215, Birkhäuser, Basel, 2004, pp. 179–198, MR 2 039 766.